3.41. Proofs, Theorems, and Inconsistency

We have stressed the marvelous correlation of the deductive system with the earlier semantic tests of validity – both approaches picking out exactly the same arguments as valid. But the parallel extends to other topics treated in semantics. The semantic approach picked out two further special families of sentences: the tautologies and the contradictions. And the inconsistency found in contradictions was then extended to *sets* of sentences. Here we show how those families of sentences are likewise picked out in by the deductive apparatus.

1. Proofs and Theorems. We recognized the following as an example of a **tautology**, or **logical truth**.[1]

P: Rex passed Chemistry

"**Either Rex passed Chemistry or he didn't pass Chemistry**."

$\mathbf{(P \vee \sim P)}$

Semantically, logical truths are true in every possible situation (every valuation). This sentence is true if Rex passed Chemistry (that is, if "P" is true); but it's also true if Rex didn't pass Chemistry (if "P" is false). Being **true regardless of the facts** about Rex and Chemistry is what makes this sentence a *logical* truth – a sentence true through its *logical form* alone. As the English example above illustrates, logical truths manage to be true independent of the facts because they are so **uninformative** about those facts. If Rex is anxious to know whether or not he passed Chemistry, the above sentence tells him nothing.

Such *truth-regardless-of-the-facts* accounts for a peculiar feature noted about logical truths and arguments[2]: *any* argument with a logical truth as its conclusion is *bound* to be valid. Since such a conclusion is always true, it provides no opportunity for a validity counterexample. A logical truth is true in and of itself, no matter which premises the argument has.

That last point is a clue to the treatment of logical truths in deductive terms. Taking a logical truth as the conclusion of an argument which is valid regardless of

[1] In Section 3.14, "*Semantic Concepts*".
[2] In Section 3.16, "*Features of Validity*".

what its premises are, we say that the deduction of such an “argument” can be completed without appeal to the premises. That is: a logical truth is the “conclusion” of an argument deducible **without premises**. Logical truths live up to their name by being deducible from the bare deductive apparatus alone – the inference rules and ID – without appeal to any premises.

A deduction with no premises is called a **proof**. And any sentence have such a proof is called a **theorem** of the deductive system.

We can construct a proof of the logical truth “(P ∨ ~P),” showing that it is a theorem of our deductive system. (We do the proofs here twice over – once without appeal to DeMorgan’s Law, once using it – to illustrate the economy DM brings.)

Without Demorgan’s Law

		~~Get~~: (P ∨ ~P)
1.	~(P ∨ ~P)	AID
		~~Get~~ P
2.	~P	AID
3.	(P ∨ ~P)	2, ∨+
4.	~(P ∨ ~P)	1, R
5.	P	2, 3, 4, ID
6.	(P ∨ ~P)	5, ∨+
7.	(P ∨ ~P)	1, 6, ID

With Demorgan’s Law

		~~Get~~: (P ∨ ~P)
1.	~(P ∨ ~P)	AID
2.	(P ∧ ~P)	1, DM
3.	P	2, ∧–
4.	~P	2, ∧–
5.	(P ∨ ~P)	1, 3, 4, ID

It is a (further) happy feature of our deductive system that its theorems – the sentences it can prove, without appeal to premises – are exactly the logical truths picked out by the semantics.[3] That means that we can show that a sentence is a (semantical) logical truth by constructing a proof of it; or that a sentence is a (provable) theorem by showing in the semantics that it's logically true.

Note that our deductive system couldn't have managed proofs until we added Indirect Deduction. For every inference rule requires at least one sentence as input; so without the AID that ID brings, the proof would never get started. Allowing proofs is therefore one more benefit to adding ID to our deductive system.

2. Inconsistent Sentences, and Sets of Sentences. The deductive system can show that a single sentence is **inconsistent** – a **contradiction** – in a number of ways. The **first approach**: since the negation of a contradiction is a logical truth, to show that a sentence is a contradiction it suffices to prove its negation.[4]

A **second approach** is to deduce from the sentence in question a known contradiction – say, a sentence of the form "(● ∧ ~●)". For as single sentences go, only a contradiction will validly entail a contradiction.[5] So if we can deduce, e.g., "(P ∧ ~P)" from a sentence, we know that sentence is a contradiction.

(Note that "(P ∧ ~P)" is just the one-sentence counterpart to the sort of opposite pair that closes an ID box. But with ID that is quarantined within the ID, never making it outside the ID box. In this contradiction test, by contrast, the contradictory conclusion at the end of the test does appear outside any ID box. Indeed, this test need not even appeal to ID.)

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[3] That **all** the logical truths of semantics are theorems of the deductive system, is referred to as the **completeness** of the deductive system. That **only** the logical truths are theorems is referred to as the **soundness** of the deductive system. While the soundness and completeness of the deductive system can be proven (in a variety of ways), such a metalogical proof lies outside the scope of this discussion.

[4] Thanks to the soundness and completeness of the deductive system, we know that (i) every contradiction has a provable negation, and (ii) only a contradiction has a provable negation. For (i): if some contradiction had an unprovable negation, that negation would be a logical truth which was unprovable. But the completeness of the deductive system rules that out. For (ii): if some non-contradictory sentence had a provable negation, then that sentence, as non-contradictory, would be true in some valuation, meaning its negation was false in that valuation, hence not a logical truth. In that case we would have a proof of a sentence which is not logically true. But the soundness of the deductive system rules that out.

[5] As noted in Section 3.16. "*Features of Validity*".

For example, we can show deductively that “~(P ∨ ~P)” is a contradiction, using either approach.

First Approach

		Get: ~~(P ∨ ~P)
1.	~~~(P ∨ ~P)	AID
2.	~(P ∨ ~P)	1, ~–
3.	(~P ∧ ~~P)	2, DM
4.	~P	3, ∧–
5.	~~P	3, ∧–
6.	~~(P ∨ ~P)	1, 4, 5, ID

Second Approach

1.	~(P ∨ ~P)	Get: (P ∧ ~P)
2.	(~P ∧ ~~P)	1, DM
3.	~P	2, ∧–
4.	~~P	2, ∧–
5.	P	4, ~–
6.	(P ∧ ~P)	3, 5, ∧+

There would thus seem little to recommend one approach over the other, as a demonstration of contradiction.

But the second approach has the advantage of scaling up, to testing **sets of sentences** for inconsistency. For while no natural candidate suggests itself for a negation of a *set* of sentences (as would be needed, along the first approach), deducing a contradiction from a set of sentences (along the second approach) is straightforward.

Summary: Theorems and Inconsistency

- A deduction of a sentence without use of premises is a **proof** of that sentence.
- A sentence which is provable is a **theorem**. Every theorem of this deductive system is a (semantic) **logical truth**.
- A sentence is a **contradiction** (is **inconsistent**) if (and only if) there is a proof of the negation of that sentence.
- A set of (one or more) sentences is **inconsistent** if a contradiction – for example, a sentence of the form "(● ∧ ~●)" – is deducible from those sentences.